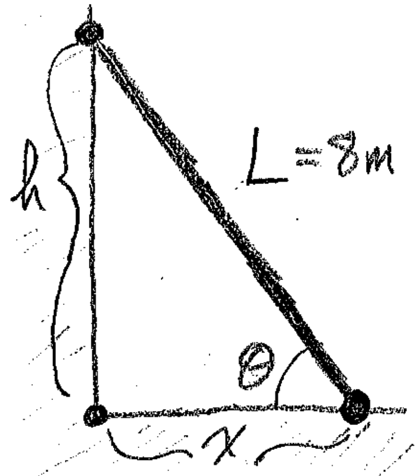
Related Rates: The Ladder Problem



Suppose a ladder of length $L = 8m$ leans against a wall, elevated at angle $\theta = 60°$. If the top of the ladder against the wall falls at a constant speed of $v = 2m/s$, how fast is the bottom of the ladder moving when the top of the ladder is four meters above the ground? (You may use that when this happens, $\frac{d\theta}{dt} = -\frac{3}{4}$.)

Solution:

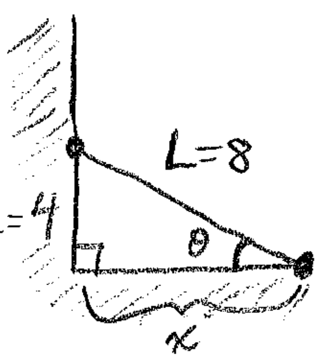
- We need to relate h and x in the picture.
  By trigonometry, $\tan\theta = \frac{h}{x}$. Thus, $x\tan\theta = h$.

- Next, we need to (implicitly) differentiate with respect to time:

$$\frac{d}{dt}(x\tan\theta) = \frac{d}{dt}(h)$$

$$\frac{dx}{dt}\tan\theta + x\frac{d}{dt}(\tan\theta) = \frac{dh}{dt}$$

$$\frac{dx}{dt}\tan\theta + x\sec^2\theta\,\frac{d\theta}{dt} = \frac{dh}{dt} \quad (*)$$



- Now, let's find the values of these numbers when $h = 4$ (i.e., the ladder's top is four meters above the ground.) From the picture, $\sin\theta = \frac{4}{8} = \frac{1}{2}$. That means $\theta = 30°$.

– We can also find x either using $8^2 = 4^2 + x^2$ or $\cos 30° = \frac{x}{8}$.
  We find that $x = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$ (or $x = \sqrt{64-16} = \sqrt{48} = 4\sqrt{3}$).

– We already know that $\frac{dh}{dt} = -2$ (since it falls <u>down</u>.)

- Note that we are also given $\frac{d\theta}{dt} = -\frac{3}{4}$.

(Technically we can solve for this, but it's more work and overcomplicates the problem.)

(Also, $\theta = 60°$ initially was extraneous, except perhaps in drawing the first picture.)

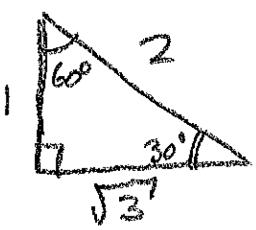
- Let's solve the equation $(*)$ for $\frac{dx}{dt}$:

$$\tan\theta \frac{dx}{dt} = \frac{dh}{dt} - x\sec^2\theta \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{1}{\tan\theta}\left(\frac{dh}{dt} - x\sec^2\theta\frac{d\theta}{dt}\right)$$

- We now plug in the values we found:

$$\frac{dx}{dt} = \frac{1}{\tan 30°}\left(-2 - 4\sqrt{3}\,(\sec 30°)^2\left(-\frac{3}{4}\right)\right)$$



$$= \frac{1}{\frac{1}{\sqrt{3}}}\left(-2 + 4\sqrt{3}\cdot\left(\frac{2}{\sqrt{3}}\right)^2\cdot\frac{3}{4}\right)$$

$$= \sqrt{3}\left(-2 + 3\sqrt{3}\cdot\frac{4}{3}\right)$$

$$= -2\sqrt{3} + (\sqrt{3})^2\cdot 4$$

$$= -2\sqrt{3} + 12$$

$$\approx 8.54$$

- Thus, the bottom of the ladder moves away from the wall at a speed of 8.54 m/s when the top is four meters above the ground.