Rainbows, Quantum Billiards, and the Birth of Reflections

Stokes Phenomenon Exemplified

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Abstract

In this talk...

- What is the Stokes Phenomenon?

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- Example One: Rainbows/The Airy function

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- Example Two: Birth of Reflections/Helmholtz Equation
- Example Three: Quantum Billiards & Weyl Expansions

Preliminaries

Asymptotic Expansions, in the sense of Poincaré

The following are equivalent. As $z \to \infty$:

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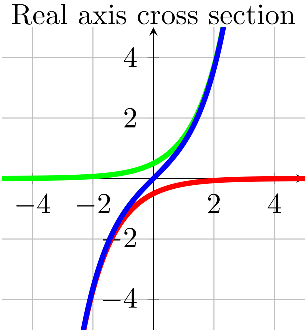

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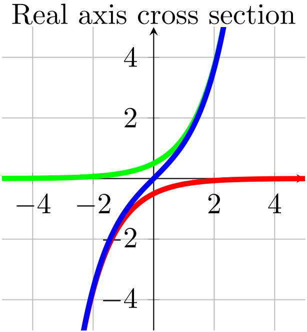

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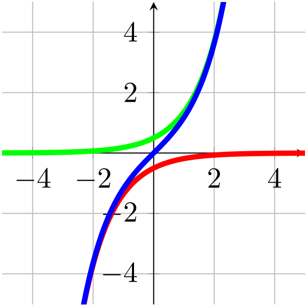
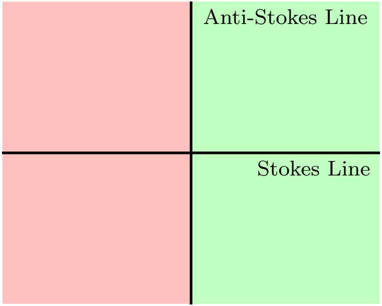
Real axis cross section



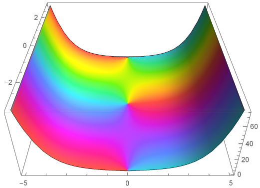
Stokes regions in $\mathbb{C}$



Sinh in the Complex Plane

Observe the change in behavior across the imaginary axis.

$$f(z) = \sinh(z)$$



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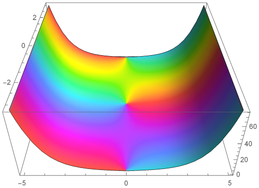
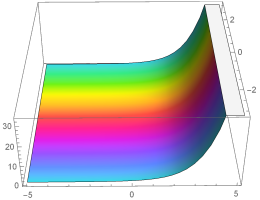


$$f(z) = \frac{1}{2}e^{z}$$



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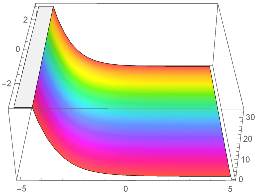

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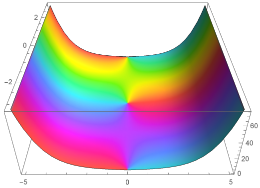

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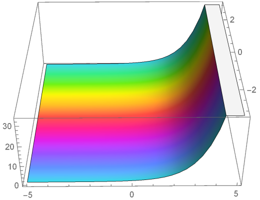

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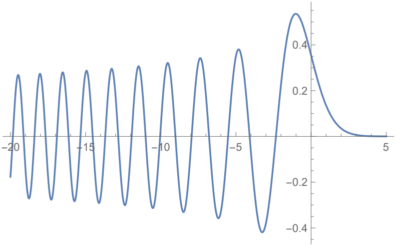
R. B. Dingle's Description

At a certain phase drawn in the complex plane as a "Stokes ray", an "associated function" appears, disappears or changes its numerical multiplier."

Supernumerary Rainbow

The Airy Model

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt$$



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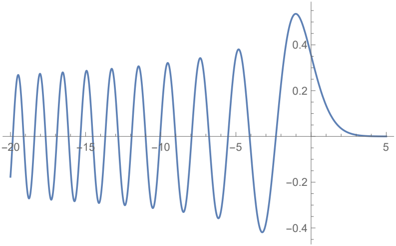

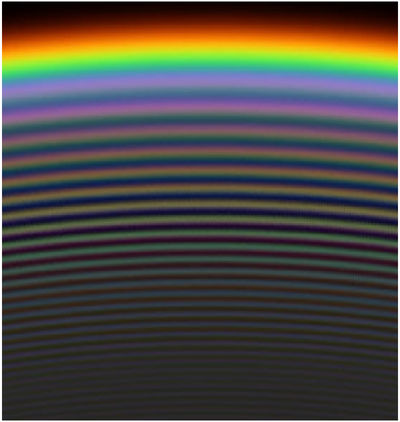

Stokes' Work

Here, U denotes a solution to the Airy equation, $\frac{d^2U}{dn^2} + \frac{n}{3}U = 0$.

$$U = An^{-\frac{1}{4}}\,\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1 - \frac{1.5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}} + \frac{1.5.7.11}{1.2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2 - \frac{1.5.7.11.13.17}{1.2.3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3 + \ldots\right\}. \quad . \quad . \quad (14)$$

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Secondly, suppose n negative, and equal to $-n'$. Then, writing $-n'$ for n in (14), and changing the arbitrary constant, and the sign of the radical, we get

$$U = Cn'^{-\frac{1}{4}}\epsilon^{-\frac{2}{3}\sqrt{\frac{n'^3}{3}}}\left\{1 - \frac{1.5}{1.16\,(3n^3)^{\frac{1}{2}}} + \frac{1.5.7.11}{1.2.16.3n^3} - \ldots\ldots\right\} \quad . \quad . \quad . \quad . \quad (17)$$

Stokes Behavior of the Airy Function

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_\gamma e^{i(zt-\frac{1}{3}t^3)}dt, \quad \mathrm{Im}(\gamma) = (-\infty e^{-\frac{2}{3}i\pi}, 0] \cup [0, \infty)$$

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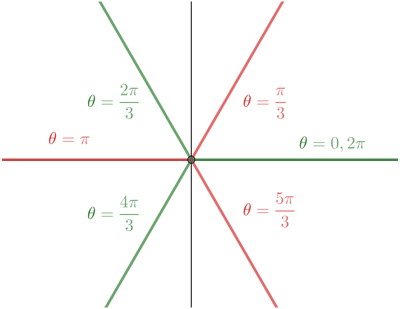
The Stokes behavior of the Airy function:



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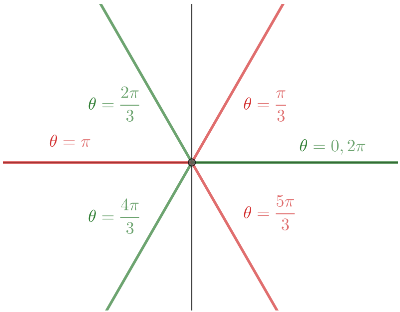

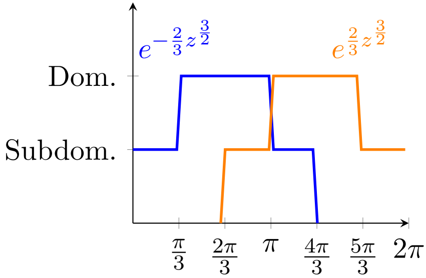

Airy Function Asymptotics

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

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$$\begin{aligned} \mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \Bigg(& \sin(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\zeta^{2n}} \\ & - \cos(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\zeta^{2n+1}} \Bigg) \qquad |\arg(z)| < \frac{2}{3}\pi \end{aligned}$$

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Notation:

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_n = \frac{\Gamma(3n + \frac{1}{2})}{54^n n! \Gamma(n + \frac{1}{2})} = \frac{(2n+1)(2n+3)...(6n-1)}{216^n n!}$$

Asymptotics on the Real Line

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} & x > 0 \\ \frac{1}{\sqrt{\pi}} (-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right) & x < 0 \end{cases}$$

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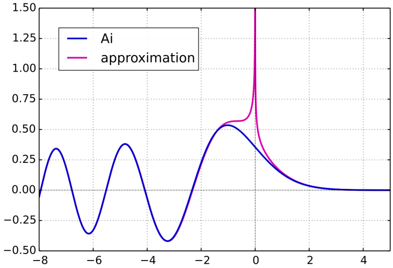
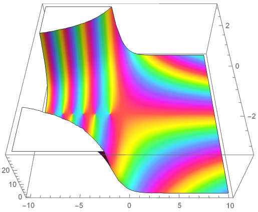


Airy Function in the Complex Plane

Complex plots of the approximations and where they agree.

$$\mathrm{Ai}(z)$$

(Entire)



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$\mathrm{Ai}(z)$	$\dfrac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{\frac{3}{2}}}$
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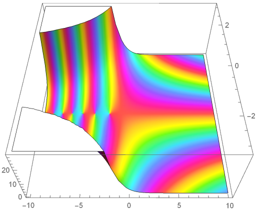	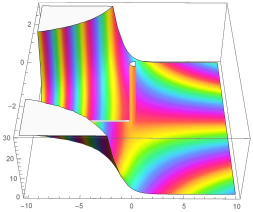

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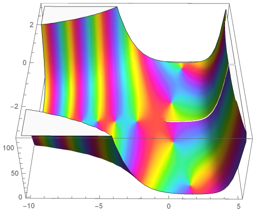
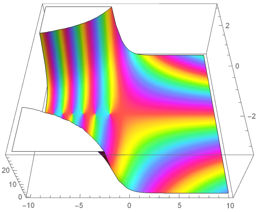
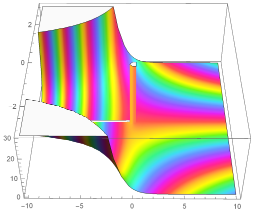
$\dfrac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right)$	$\mathrm{Ai}(z)$	$\dfrac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$
$\lvert\arg(-z)\rvert < \dfrac{2\pi}{3}$	(Entire)	$\lvert\arg(z)\rvert < \pi$







Example Two: WKB Solution to a Helmholtz Equation

Consider this one-dimensional Helmholtz Equation:

$$\frac{d^2u}{dz^2}(z) = k^2 R^2(z) u(z)$$

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Exponentially weak reflections arise:

M. Berry

We are able to answer this question [i.e. where and how does the reflected wave arise on the x-axis], because the birth of a reflection is simply the switching-on of a subdominant multiplier.

Stokes Lines for the Equation

Stokes lines arise from zeroes of R, say z_j, wherein:

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M. Berry

Stokes lines lie at the heart of the asymptotics of [this equation.] They are the locus of greatest disparity between the dominant and subdominant fundamental phase-integral approximate solutions attached to z_j:

$$u_{\pm} \approx \exp(\pm k w(z))/R^{\frac{1}{2}}(z).$$

Behavior of the Waves

Dominant u_+ corresponds to the incident wave,
Subdominant u_- to the reflected wave.

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WKB Approximate Solution:

$$u(z) \approx a_+(z)u_+(z) + a_-(z)u_-(z)$$

Across a Stokes line, the multiplier a_- jumps by ia_+.

Example Three: Quantum Billiards

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Consider the Dirichlet eigenvalue problem:

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Regularized Resolvent:

$$g(s) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{E_n + s^2} - \frac{\mathscr{A}}{4\pi}\log\left(\frac{E_N}{s^2}\right)\right)$$

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Weyl Expansion:

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One may truncate to the least term, obtaining an exponentially small remainder.

$$g(s) = \sum_{r=1}^{r^*} \frac{c_r}{s^r} + R(s)$$

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Across Stokes lines, the remainder changes behavior becoming oscillatory, then large.

$$e^{-s} \quad \longrightarrow \quad e^{-is} \quad \longrightarrow \quad e^{s}, \quad s \in \mathbb{R}^+$$

Next Time: Segue into Resurgence

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- Quantum Billiards Examples

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- Spectral Resurgence

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- Spectral Resurgence
- Deducing Stokes behavior from expansions

Next Time: Segue into Resurgence

- Quantum Billiards Examples
- Spectral Resurgence
- Deducing Stokes behavior from expansions
- And more!

Historical References

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- Geek3, Airy Ai Asymptotic [SVG, converted to PNG], 7 Feb. 2015, Wikimedia Commons. License: CC BY 3.0.