Rainbows, Quantum Billiards, and the Birth of Reflections

Segue into Resurgence

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- Across Stokes lines, non-perturbative terms appear.
- The phenomenon occurs with: the Airy function, WKB solutions, and Weyl expansions.

Abstract

In this talk...

- How does one find such non-perturbative terms?

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- Connections to spectral or fractal geometry?

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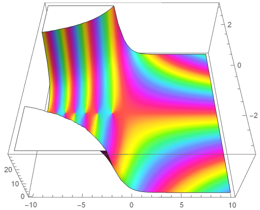
- In depth look at the Airy series
- Contrasting examples: Harmonic Oscillator & Particle on a Ring
- Current investigations

Airy Function Recap

The Airy function and two different asymptotic expansions (to first order.)

$$\mathrm{Ai}(z)$$

(Entire)



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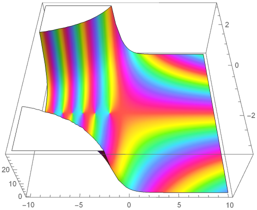
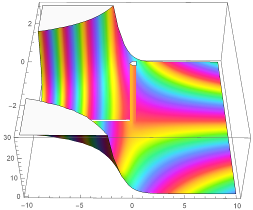


$$\frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$$

$$|\arg(z)| < \pi$$



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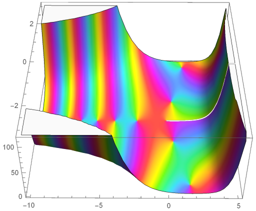
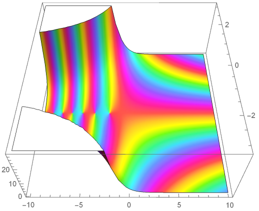
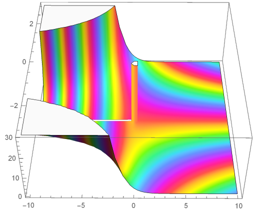
$$\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) \qquad \mathrm{Ai}(z) \qquad \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$$

$$|\arg(-z)| < \frac{2\pi}{3} \qquad \text{(Entire)} \qquad |\arg(z)| < \pi$$







Airy Asymptotics Recap

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

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$$\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \Bigg(\sin(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\zeta^{2n}} - \cos(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\zeta^{2n+1}} \Bigg) \qquad |\arg(z)| < \frac{2}{3}\pi$$

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Notation:

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_n = \frac{\Gamma(3n + \frac{1}{2})}{54^n \Gamma(n+1)\Gamma(n + \frac{1}{2})}$$

Airy Function Expansion

The Airy function is governed by the asymptotic expansion:

$$\varphi_{\mathrm{Ai}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} \frac{1}{z^n}$$

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This is the same formula as previously seen, since:

$$a_n = \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} = \left(-\frac{2}{3}\right)^{-n} \frac{\Gamma(3n+\frac{1}{2})}{54^n\Gamma(n+1)\Gamma(n+\frac{1}{2})}$$

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More remarks:

- φ_{Ai} is factorially divergent (of Gevrey-class one.)

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More remarks:

- φ_{Ai} is factorially divergent (of Gevrey-class one.)
- $z = k^{\frac{3}{2}}$ is a natural change of variables for ensuing resummation.

Airy Series: Borel Summation

- The minor of φ_{Ai} is its (formal) Borel transform, forgetting the constant term:

$$\tilde{\varphi}_{\text{Ai}} := \mathcal{B}[\varphi_{\text{Ai}}] = \sum_{n=1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$$

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- $\tilde{\varphi}_{\text{Ai}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \{0, -\frac{4}{3}\}$
- For any direction θ not along the negative real axis, the following converges for $\text{Re}(ze^{i\theta}) > 0$:

$$S_\theta \varphi_{\text{Ai}}(z) := a_0 + \mathcal{L}_\theta \mathcal{B}[\varphi_{\text{Ai}}](z) = a_0 + \int_0^{\infty e^{i\theta}} \tilde{\varphi}_{\text{Ai}}(\zeta) e^{-z\zeta} d\zeta$$

A Borel Resummed Expansion

Where before:

$$\mathrm{Ai}(k) \sim \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

We now have:

$$\mathrm{Ai}(k) = \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} S_0 \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

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This resummation is valid for $|\arg(k)| < \frac{\pi}{3}$, $|k| > 0$.

One can rotate the direction of summation for new regions of validity.

Contours near the Singularity at $-\frac{4}{3}$

Courtesy of Delabaere:

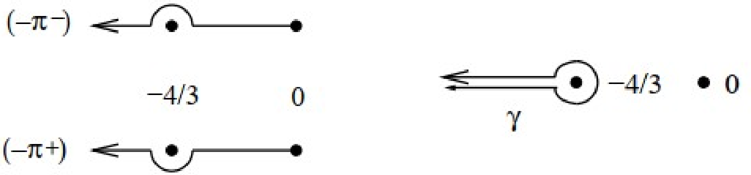

FIGURE 2. Right and left Borel-resummation.

One can compares right and left-resummations, since

$$\text{(4)} \qquad s_{-\pi^-}\varphi_{Ai}(z) = s_{-\pi^+}\varphi_{Ai}(z) + \int_\gamma \widetilde{\varphi_{Ai}}(\zeta)e^{-z\zeta}\,d\zeta$$

Alien Calculus & Behavior across the Singularity

The Hankel contour γ can be expressed using the so-called alien derivative:

$$\int_{\gamma} \tilde{\varphi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} d\zeta = e^{+\frac{4}{3}z} S_{-\pi} \left(\Delta^{z}_{-\frac{4}{3}} \varphi_{\mathrm{Ai}} \right)(z)$$

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In this case,

$$\Delta^z_{-\frac{4}{3}}\varphi_{\mathrm{Ai}} = -i\varphi_{\mathrm{Bi}}, \quad \varphi_{\mathrm{Bi}}(z) := \sum_{n=0}^{\infty}(-1)^n\frac{a_n}{z^n}$$

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φ_{Bi} is also Gevrey-1 and its minor $\tilde{\varphi}_{\mathrm{Bi}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \left\{0, +\frac{4}{3}\right\}$.

Airy Function on the Negative Real Line

Deducing the behavior Ai for negative real inputs.

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Once can rewrite the LHS as the resummed version of the second expansion we saw previously.

Zeroes and the Real Line

This expression can be rewritten to calculate the zeroes on the Airy function.

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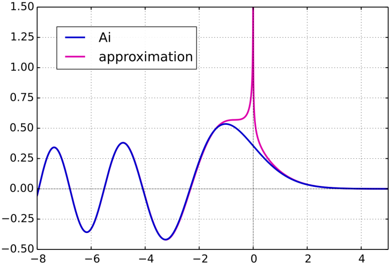

The Upshot & Transition

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- The Stokes phenomenon related to the Airy function can be analyzed using methods from resurgent analysis and Borel resummation.
- Follow up: when do we expect to see such behavior?

Spectral Functions

Counting Function:

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Spectral Resolvent (regularized):

$$g(s) = \int_0^{\infty} e^{-s^2 t} \left(K(t) - \frac{a_0}{t} \right) dt$$

Harmonic Oscillator

A quantum harmonic oscillator has a Hamiltonian of the form:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

The energy levels are given by:

$$E_n = \left(n + \frac{1}{2}\right)\omega, \quad n \in \mathbb{N}$$

Heat Kernel:

$$K(t) := \sum_{n=1}^{\infty} e^{-E_n t} = \frac{1}{2\sinh\left(\frac{1}{2}\omega t\right)}$$

Harmonic Oscillator: Spectral Resolvent

The resolvent can be written as the formal series:

$$g(s) = -\frac{2}{\omega} \sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{\infty} (-1)^k (2k-1)! \left(\frac{\omega}{2\pi s^2 m} \right)^{2k}$$

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Remark: the poles occur at $g(i\sqrt{E_n})$.

$$-\frac{2}{\omega}\sum_{m=1}^{\infty}(-1)^m R_m(i\sqrt{E}) \approx \frac{\pi}{\omega}\left(i + \tan\left(\frac{\pi E}{\omega}\right)\right)$$

Free Particle on a Ring

A particle on a ring solves the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{R^2\partial\theta^2}\psi = E\psi$$

The energy levels are given by:

$$E_n = \pi n^2, \quad n \in \mathbb{Z}$$

Heat Kernel:

$$K(t) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi t} = \frac{1}{\sqrt{t}}\left(1 + 2\sum_{m=1}^{\infty} e^{-\frac{m^2\pi}{t}}\right)$$

Particle on a Ring: Spectral Resolvent

The resolvent can be defined in this case without regularization, due to convergence. In particular:

$$g(s) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2\pi + s^2} = \frac{\sqrt{\pi}}{s} \coth(s\sqrt{\pi})$$

$$= \frac{\sqrt{\pi}}{s} \left(1 + 2 \sum_{m=1}^{\infty} e^{-2sm\sqrt{\pi}} \right)$$

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In this situation, there is no resurgence/Stokes phenomenon.

The semi-classical approximations for the propagator (the trace of K) and the energy Green's function are exact.

Contrasted Examples & What's Next?

- The examples show that non-exactness and divergence are connected to resurgency/ Stokes phenomena.

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- This gives clues as to where to such phenomena would appear in explicit formulae.
- Based on some results about lacunary series and natural boundaries, we now ask the question...

Question for Next Time...

Is is possible to glimpse behind the screen?

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Inspired by results in "Behavior of Lacunary Series at the Natural Boundary" by Costin and Huang.

Historical References

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Images

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