A First Introduction to Resurgence

Part 2

Will Hoffer

University of California Riverside

whoff003@ucr.edu

May 27, 2020

Review & Overview

From Part 1:

- What is an asymptotic expansion?
- What is Borel Summation?
- Example: $\sum_{n=0}^{\infty}(-1)^n n! z^{-(n+1)}$

Review & Overview

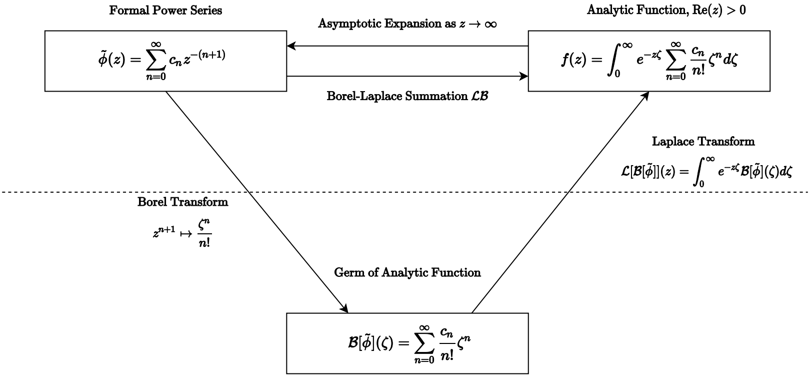
From Part 1:

- What is an asymptotic expansion?
- What is Borel Summation?
- Example: $\sum_{n=0}^{\infty}(-1)^n n! z^{-(n+1)}$

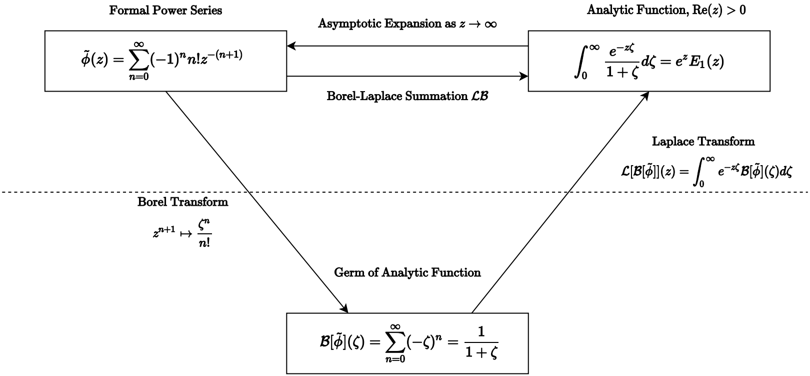
For Part 2:

- What happens when singularities occur in the Borel plane?
- What is resurgence?
- What has been done with resurgent techniques?

Borel Summation Schematic



Borel Summation Example



Borel Summation & Singularities

Example with Singularity:

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} n! z^{-(n+1)}$$

$$\mathcal{B}[\tilde{\phi}](\zeta) = \sum_{n=0}^{\infty} \zeta^n = \frac{1}{1-\zeta}$$

The Laplace transform along $\mathbb{R}^+$ encounters $+1$.

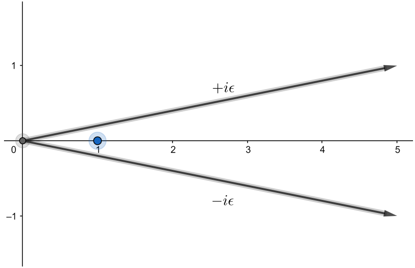
Directional Laplace Transforms

Directional Laplace Transform

$$\mathcal{L}^{\theta}[f](z) = \int_0^{\infty e^{i\theta}} e^{-z\zeta} f(\zeta) d\zeta$$

Ambiguity:

Above or below singularity?



Stokes Phenomenon

Denote by S^+ (resp. S^-) the operation $\mathcal{L}^{+i\epsilon}$ (resp. $\mathcal{L}^{-i\epsilon}$)

Discrepancy:

$$(S^+ - S^-)[\tilde{\phi}](z) = 2\pi i e^{-z}$$

Remark:

$$\tilde{\phi}' + \tilde{\phi} = \frac{1}{z}$$

$$\phi' + \phi = 0$$

The line $\theta = 0$ is a Stokes line.

Algebra of Resurgent Functions I

Definition

Resurgent functions are the formal power series, arising from a Borel transform, which are germs of analytic functions.

- The convolutive model expresses them as the Borel-transformed series.
- The multiplicative model expresses them as the image under the Laplace transform.

Algebra of Resurgent Functions II

Convolution:

$$\hat{\mathcal{H}}(\mathcal{R}) = \{\text{Analytic germs at the origin}\}$$

Multiplication:

$$\widetilde{\mathcal{H}}(\mathcal{R}) = \mathcal{B}^{-1}(\widehat{\mathcal{H}}(\mathcal{R})) \subset z^{-1}\mathbb{C}[[z^{-1}]]$$

Adjoining the convolutive unit, $\delta = \mathcal{B}[1]$

$$\widehat{\mathscr{R}} = \mathbb{C}\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$$
$$\widetilde{\mathscr{R}} = \mathcal{B}^{-1}(\hat{\mathscr{R}})$$

Stokes Automorphism

From lateral Borel summations $\mathcal{S}_\theta^\pm$, define a map $\mathfrak{S}_\theta$ via:

$$\mathcal{S}_{\theta+} = \mathcal{S}_\theta^- \circ \mathfrak{S}_\theta = \mathcal{S}_\theta^- \circ (\mathrm{Id} - \mathrm{Discont}_\theta)$$
$$\mathcal{S}_{\theta+} - \mathcal{S}_\theta^- = -\mathcal{S}_\theta^- \circ \mathrm{Discont}_\theta$$

$\mathfrak{S}_\theta$ is an automorphism of $\widehat{\mathscr{R}}$.

Alien Derivative I

Definition

The alien derivative (French: *étranger*) Δ_ω is given by:

$$\mathfrak{S}_\theta = \exp\left(\sum_{\omega\in\Gamma_\theta} e^{-\omega z}\Delta_\omega\right)$$

Δ_ω is a derivation of the algebra of resurgent functions:

$$\Delta_\omega(\hat{\phi}_1 * \hat{\phi}_2) = (\Delta_\omega\hat{\phi}_1) * \hat{\phi}_2 + \hat{\phi}_1 * (\Delta_\omega\hat{\phi}_2)$$
$$\Delta_\omega(\tilde{\phi}_1 \cdot \tilde{\phi}_2) = (\Delta_\omega\tilde{\phi}_1)\hat{\phi}_2 + \hat{\phi}_1(\Delta_\omega\tilde{\phi}_2)$$

Alien Derivative II

Given a simple resurgent function

$$\hat{\phi}(\zeta) = \frac{\alpha}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i}\hat{\Phi}(\zeta - \omega)\log(\zeta - \omega)$$

The alien derivative satisfies

$$\Delta_\omega \hat{\phi}(\zeta) = \alpha\delta + \hat{\Phi}(\zeta)$$

The alien derivative is connected to the ordinary derivative via Écalle's bridge equation.

Important of the Alien Derivative

Écalle lists the following as useful features of Δ_ω:

I Derivation of the algebra

II Measure singularities at/over the point ω

III Connect behavior near the origin to other singular points ω

Important of the Alien Derivative

Écalle lists the following as useful features of Δ_ω:

I Derivation of the algebra

II Measure singularities at/over the point ω

III Connect behavior near the origin to other singular points ω

Écalle on point III:

They enable us to describe, by means of so-called resurgence equations of the form $E_\omega(\overset{\triangledown}{\phi}, \Delta_\omega \overset{\triangledown}{\phi}) \equiv 0$, the very close connection which usually exists between the behavior of $\hat{\phi}(\zeta)$ near $0_\bullet$ and near its other singular points ω.

Important of the Alien Derivative

Écalle lists the following as useful features of Δ_ω:

I Derivation of the algebra

II Measure singularities at/over the point ω

III Connect behavior near the origin to other singular points ω

Écalle on point III:

They enable us to describe, by means of so-called resurgence equations of the form $E_\omega(\overset{\triangledown}{\phi}, \Delta_\omega \overset{\triangledown}{\phi}) \equiv 0$, the very close connection which usually exists between the behavior of $\hat{\phi}(\zeta)$ near $0_\bullet$ and near its other singular points ω.

This self-reproduction property is an outstanding feature of all resurgent functions of natural origin (their birth-mark, as it were!) and it is precisely what the label "resurgence" (bestowed somewhat promiscuously on the whole algebra $\mathrm{R\overset{\triangledown}{E}S}$) is meant to convey.

Further Elements of the Theory

Median summation

- Construct an unambiguous average across Stokes lines using $\mathfrak{S}_\theta$ for which real series yield real sums.

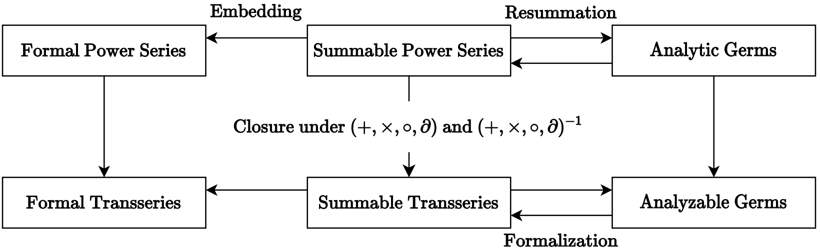
Transseries

- Description of the series in correspondence with resurgent functions.

Differential Equations

- The resurgence phenomenon is largely focused on the emergence of differential equation structure from a formal series *in and of itself*

Transseries & Analyzability



Some Notable Applications

Dulac's Conjecture

- On finiteness of limit cycles; related to Hilbert's 16^{th} problem
- Écalle's proof relies on resurgent functions

Quantum Field Theory

- Exponentially small, non-analytic corrections to perturbative expansions ("instantons")
- Potential to recovering nonperturbative effects through resurgence of a perturbative expansion

More Applications

- Normal forms of dynamical systems
- Gauge theory of singular connections
- Quantization of symplectic and Poisson manifolds
- Floer homology and Fukaya categories
- Knot invariants
- Wall-crossing and stability conditions in algebraic geometry
- Spectral networks
- WKB approximation in quantum mechanics
- Non-linear differential equations and asymptotics

Sources

- Costin, *Asymptotics and Borel Summability*, Chapman & Hall/CRC Press, 2009.
- Dorigoni, "An Introduction to Resurgence, Trans-Series and Alien Calculus," arXiv, 2015.
- Écalle, "Six Lectures on Transseries, Analysable Functions and the Constructive Proof of Dulac's Conjecture," *Bifurcations and Periodic Orbits of Vector Fields*, 1992.
- Pym, "Resurgence in Geometry and Physics Lectures," U. of Oxford Course, 2016.
- Sauzin, "Resurgent functions and splitting problems," arXiv, 2006.