# Tube Formulae for Generalized von Koch Fractals

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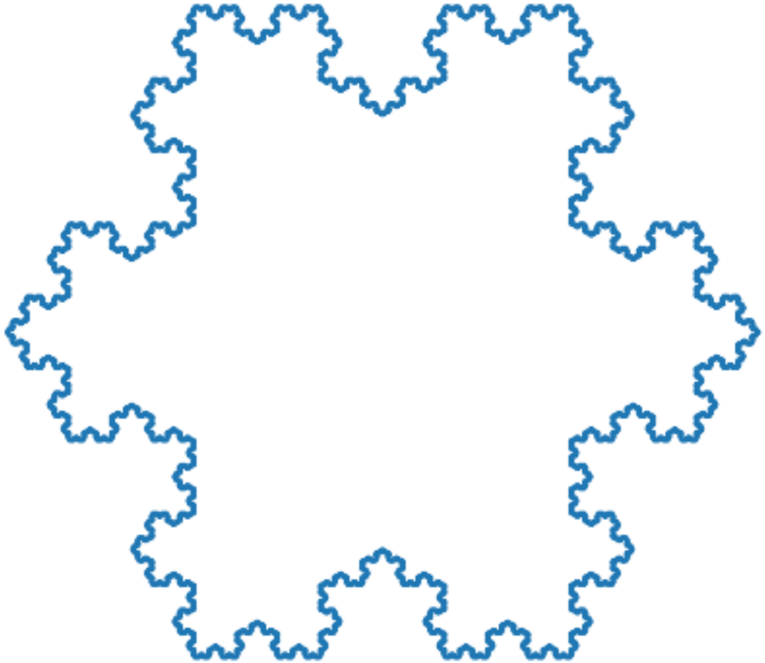
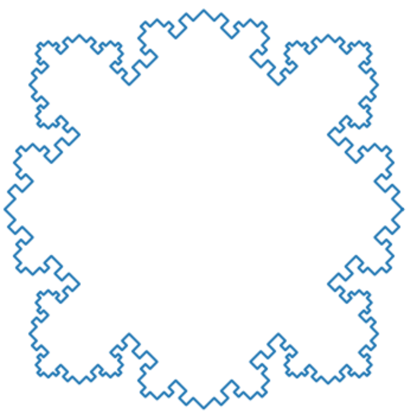
# The Von Koch Snowflake

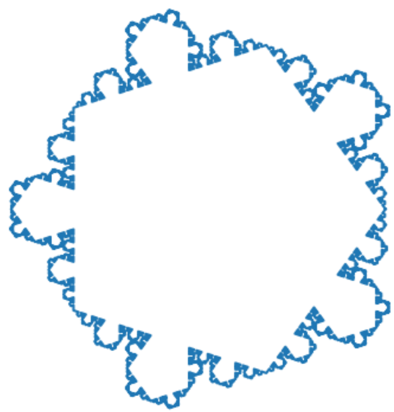


# Generalized von Koch Fractals $K_{n,r}$

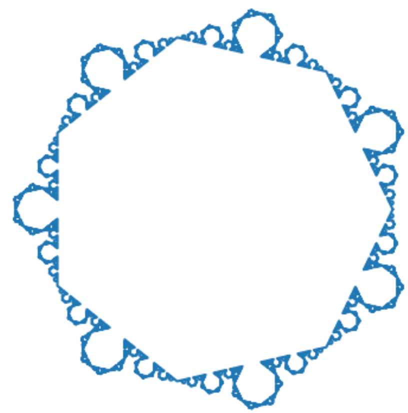
$K_{4,\frac{1}{4}}$



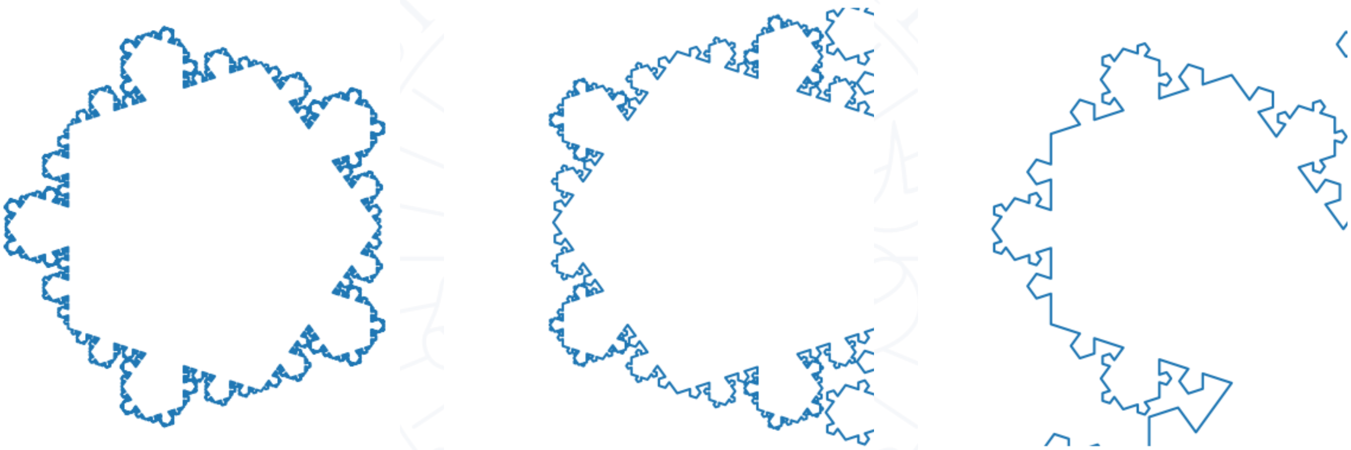
$K_{5,\frac{1}{5}}$



$K_{7,\frac{1}{8}}$



# A Closer Look at $K_{5,\frac{1}{5}}$



# Tube Formulae for $K_{n,r}$

The main goal is to study the volume of an (inner) epsilon neighborhood of the fractal set $\partial K_{n,r}$.

$$V_{K_{n,r}}(\varepsilon) = m_{\text{Leb}}(\{x \in K_{n,r} \; : \; d(x, \partial K_{n,r}) < \varepsilon\})$$

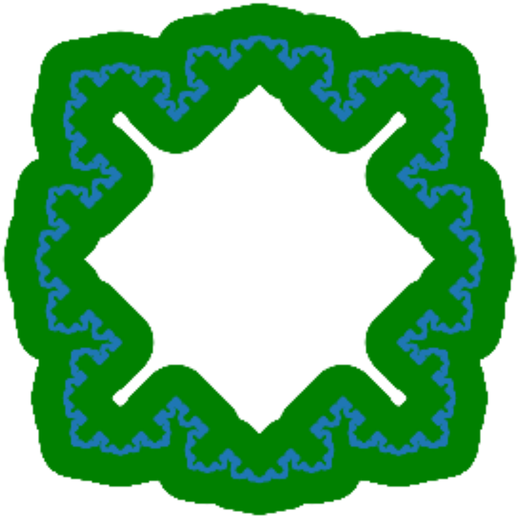


Figure: The full (not inner) epsilon neighborhood of the fractal $K_{4,\frac{1}{4}}$

# Known Results for $K_{3,\frac{1}{3}}$

Lapidus and Pearse have previously obtained an explicit tube formula for the ordinary von Koch snowflake fractal, $K_{3,\frac{1}{3}}$, which takes the form [LP06]:

$$V_{K_{3,\frac{1}{3}}}(\varepsilon) = \sum_{n\in\mathbb{Z}} \varphi_n \varepsilon^{2-(D+inp)} + \sum_{n\in\mathbb{Z}} \psi_n \varepsilon^{2-inp}$$

Here $\varphi_n$ and $\psi_n$ are constants depending only on $n$, $D = \log_3 4$, and $p = 2\pi/\log 3$.

The constants for the exact tube formula are not explicitly given, however. They depend on a function related to the error in their initial approximation.

# Tube Zeta Functions [LRŽ17]

The relative tube zeta function of a (fractal) set $\partial\Omega$ relative to $\Omega \subset \mathbb{R}^2$ is defined by:

$$\widetilde{\zeta}_\Omega(s) = \int_0^\delta t^{s-2} V_\Omega(t) \frac{dt}{t},$$

where $V_\Omega = |(\partial\Omega)_\varepsilon \cap \Omega|$ is the volume of an inner $\varepsilon$-neighborhood.

The poles of the zeta function are called the complex dimensions of $\partial\Omega$, and they describe the geometry of $\partial\Omega$.

# Explicit Tube Formula, General Results

Lapidus, Radunović, and Žubrinić have shown explict tube formulae for $V_\Omega$, expressed in terms $\widetilde{\zeta}_\Omega$.

Assume that $\widetilde{\zeta}_\Omega$ admits a meromorphic continuation to (an open, connected neighborhood of) $\mathrm{Re}(s) > \sigma_0$ and satisfies appropriate languid growth conditions. Then the tube formula satisfies [LRŽ17]:

$$V_\Omega^{[k]}(\varepsilon) = \sum_\omega \mathrm{Res}\left(\frac{\varepsilon^{2-s+k}}{(2-s+1)_k}\widetilde{\zeta}_\Omega(s);\omega\right) + O(\varepsilon^{2-\sigma_0+k})$$

Here, $V_\Omega^{[k]}$ is the $k^{\text{th}}$ antiderivative of $V_\Omega$, and the formula holds pointwise when $k$ is large enough.[1]

[1] The bound depends upon the specific exponent appearing in the languid growth conditions.

# Toward Explicit Tube formula for $K_{n,r}$

The goal of this line of research is to more concretely describe tube formulae for $V_{K_{n,r}}(\varepsilon)$.

Today, we will present how one may obtain leading order terms of the form:[2]

$$V_{K_{n,r}}(t) = \sum_{\omega} a_\omega t^{2-\omega} + O(t^{2-\varepsilon}), \quad t \to 0^+.$$

Here the sum runs over each solution $\omega$ to:

$$1 - 2\left(\frac{1-r}{2}\right)^{\omega} - (n-1)r^{\omega} = 0$$

[2] This representation may need to be interpreted distibutionally.

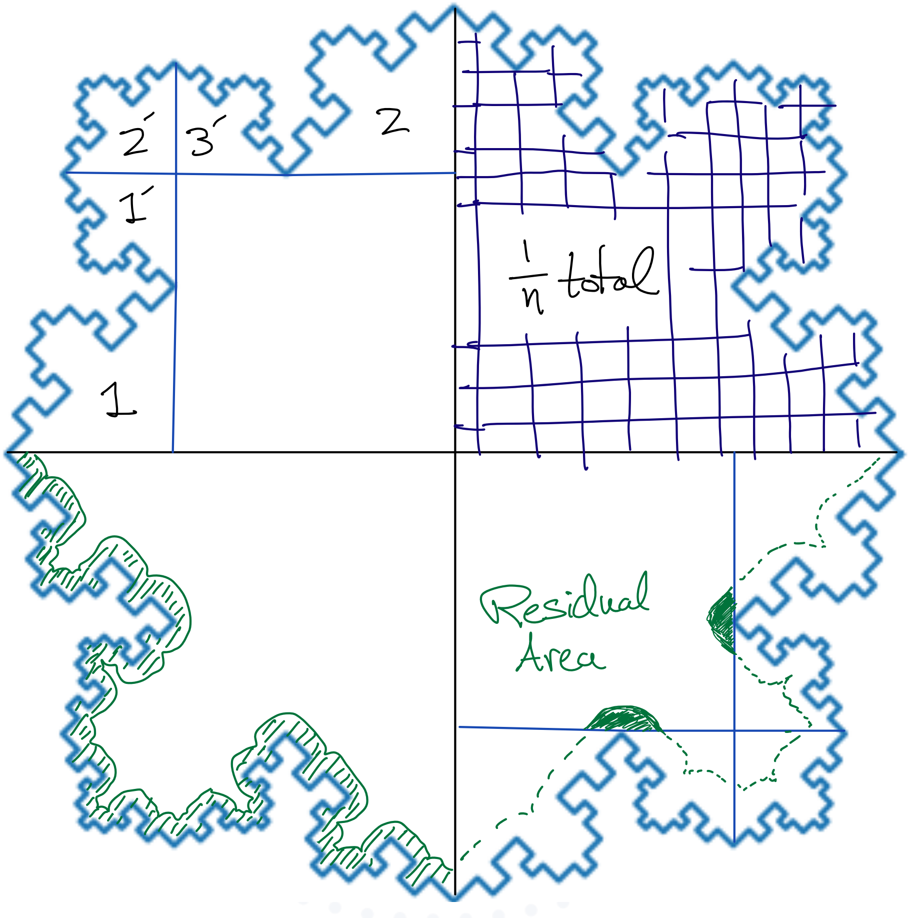
# Strategy for Analyzing $V_{K_{n,r}}(\varepsilon)$

The goal is to adapt the method used by Michiel van den Berg and collaborators to study asymptotics of heat content in fractals, in work such as [vdB51, vdBG98].

The method is to first divide $K_{n,r}$ into $n$ congruent pieces. Then, one subdivides each piece into $2 + (n - 1)$ copies of the same shape, but scaled down in size.

One obtains an approximate functional equation relating the volume of one piece to the volume of the scaled copies and a residual volume term, creating an approximate functional equation.

# Subdividing the Fractal $K_{n,r}$



# Scaled Pieces of the Subdivision

When we subdivide $K_{n,r}$, we obtain:

- 2 pieces which have been scaled by a factor of $\ell := \frac{1}{2}(1-r)$ from the two sides
- $n-1$ copies of the shape scaled by $r$ from the pieces on the $n$-gon attached in to middle

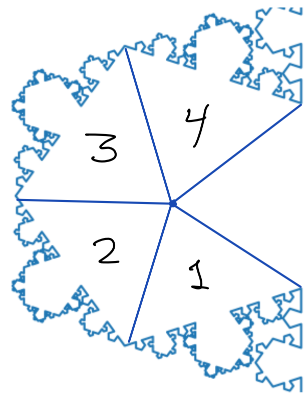



Figure: The 4 scaled copies of the subdivided $K_{5,\frac{1}{5}}$ which have been scaled by $r = \frac{1}{5}$

# Adding the Volumes from the Disjoint Pieces

We may express the total volume of the $\varepsilon$-neighborhood as a sum of the volumes contained in disjoint pieces of our partition of $K_{n,r}$.

Let us define $V_K(\varepsilon)$ to be the volume of one of the $n$ pieces, called $K$, in the initial subdivision. Since the shape has $n$-fold symmetry, this is exactly one $n^{\text{th}}$ of the total volume:

$$V_K(\varepsilon) = \frac{1}{n} V_{K_{n,r}}(\varepsilon)$$

We may then write $V_K$ as a sum of the $2 + (n-1)$ scaled pieces, together with the volume in the residual portions, called $V_{\text{Err}}$:

$$V_K(\varepsilon) = 2V_{\ell K}(\varepsilon) + (n-1)V_{rK}(\varepsilon) + V_{\text{Err}}(\varepsilon)$$

# Scaling Properties of $V_K(\varepsilon)$

Suppose we scale a shape $X$ by a factor $\lambda$. Then we have that $\varepsilon$ will scale linearly with $\lambda$ and the volume will scale quadratically with $\lambda$. Thus:

$$V_{\lambda X}(\lambda\varepsilon) = \lambda^2 V_X(\varepsilon)$$

Therefore, we have that:

$$V_{\lambda K}(\varepsilon) = \lambda^2 V_K(\varepsilon/\lambda)$$

We will use this to rewrite the previous sum of volumes coming from scaled copies of $K$ in terms of only $V_K$ (and a remainder term.)

# The Approximate Functional Equation for $V_X(\varepsilon)$

Using this scaling property, we may rewrite the equation:

$$V_K(\varepsilon) = 2V_{\ell K}(\varepsilon) + (n-1)V_{rK}(\varepsilon) + V_{\mathrm{Err}}(\varepsilon)$$

as an approximate functional equation for $V_K$.

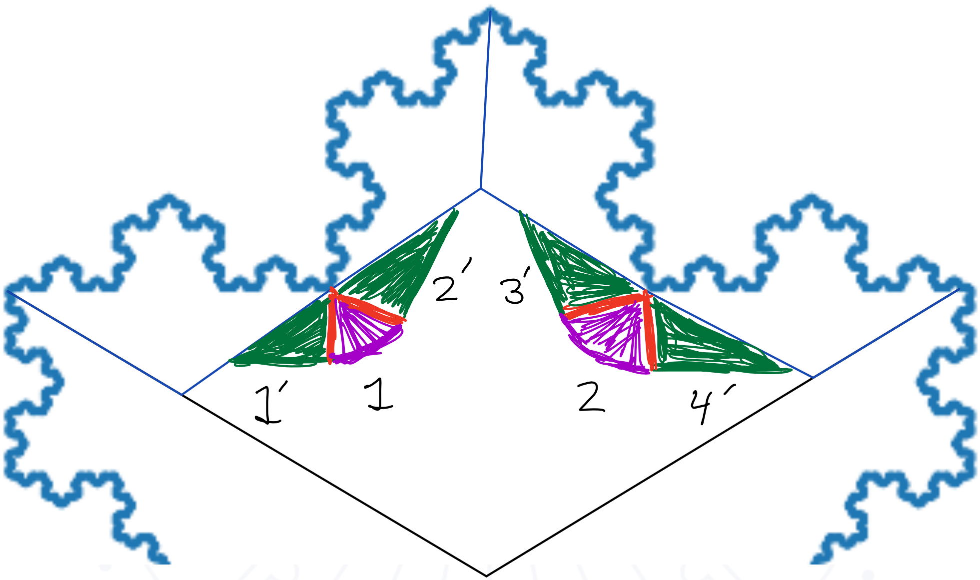
Namely, we have that $V_K$ satisfies:

**Approximate Functional Equation for $V_K$**

$$V_K(\varepsilon) = 2\ell^2 V_K(\varepsilon/\ell) + (n-1)r^2 V_K(\varepsilon/r) + V_{\mathrm{Err}}(\varepsilon)$$

# Shapes for the Remainder Volume

The volume $V_{\text{Err}}$ can be seen to arise from two types of shapes.




Thus, we may write:

$$V_{\text{Err}}(\varepsilon) = 4V_T(\varepsilon) + 2V_W(\varepsilon)$$

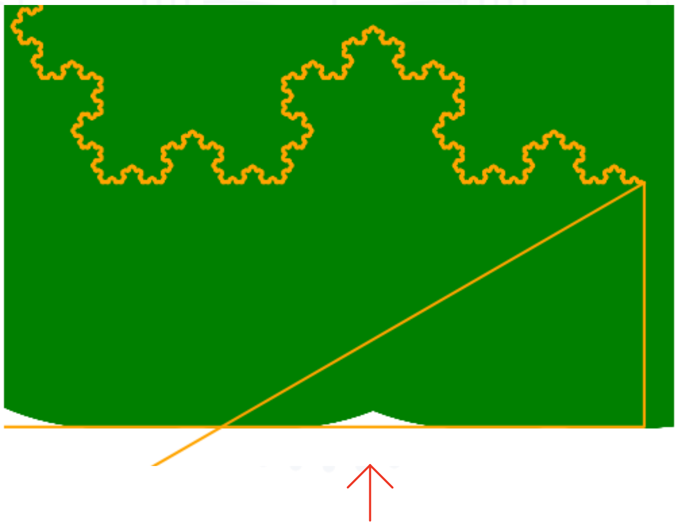
# Approximation of the Remainder

It can be shown that $V_W(\varepsilon) = 2\theta\varepsilon^2$, where $\theta = \frac{\pi}{2} - \frac{\pi}{n}$.

The other term, $V_T$, is by far the most complicated. An elementary approximation is given by:

$$V_T(\varepsilon) \approx \frac{1}{2}\varepsilon^2 \cot\theta$$

This overcounts the volume by a slight amount.



# Rewriting the Remainder Volume

Let us express $V_{\mathrm{Err}}$ in terms of a new error volume given by:

$$R(\varepsilon) = \frac{1}{2}\cot\theta\varepsilon^2 - V_T(\varepsilon)$$

It satisfies an estimate based on the largest interval of length $\ell^k$, called $L_{\max}$, in the triangle:[3]

$$0 \leq R(\varepsilon) \leq \frac{1}{2}L_{\max}^2\cot\theta = O(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

Thus we have that:

$$V_{\mathrm{Err}}(\varepsilon) = 4\left(\frac{1}{2}\cot\theta\varepsilon^2 - R(\varepsilon)\right) + 2(2\theta\varepsilon^2)$$

[3] Note that $L_{\max}$ depends on $\varepsilon$.

# Rewriting the Functional Equation

**Approximate Functional Equation (Multiplicative Form)**

$$V_K(\varepsilon) = 2\ell^2 V_K(\varepsilon/\ell) + (n-1)r^2 V_K(\varepsilon/r) + C\varepsilon^2 - 4R(\varepsilon)$$

Here, $C = (2\cot\theta + 4\theta)$.

If we change variables by $\varepsilon = e^{-x}$ and define $f(x) = V_K(e^{-x})$ and $\varphi(x) = Ce^{-2x} + 4R(e^{-x})$, we have:

**Approximate Functional Equation (Additive Form)**

$$f(x) = 2\ell^2 f(x + \log \ell) + (n-1)r^2 f(x + \log r) + \varphi(x)$$

This form can be ammenable to using renewal equation techniques, such as in [LV96].

# Functional Equation for $\widetilde{\zeta}_K$

We may now use volume functional equation to write a functional equation for the corresponding tube zeta function:

$$\widetilde{\zeta}_K(s) = \int_0^\delta t^{s-3} V_K(t)dt$$

This zeta function has the property that $\widetilde{\zeta}_{\lambda K}(s;\delta) = \widetilde{\zeta}_K(s;\delta/\lambda)$, and changing $\delta$ only changes the function by an entire holomorphic function (not affecting the poles or their residues) [LvF13]. Thus:

$$\widetilde{\zeta}_K(s;\delta) = 2\ell^s \widetilde{\zeta}_K(s;\delta/\ell) + (n-1)r^s \widetilde{\zeta}(s;\delta/r) + C\frac{\delta^s}{s} + \int_0^\delta t^{s-3} R(t)dt$$

# Solving for $\widetilde{\zeta}_K$

Letting $h(s)$ denote a (sum of) entire functions arising from relating the different values of $\delta$. Then we have:

$$\widetilde{\zeta}_K(s) = \frac{1}{1 - 2\ell^s - (n-1)r^s}\left(h(s) + C\delta^s \frac{1}{s} + \int_0^\delta t^{s-3}R(t)dt\right)$$

Since the term in the term in parenthesis is a holomorphic function in the half-plane $\mathrm{Re}(s) > 0$, we deduce that $\widetilde{\zeta}_K$ has simple poles $\omega$ for each solution to the complexified Moran equation:[4]

$$1 - 2\ell^\omega - (n-1)r^\omega = 0$$

[4] Provided that $\mathrm{Re}(\omega) > 0$.

# Leading Order Asymptotics of $V_{K_{n,r}}$

This information and polynomial growth estimates[5] for $\widetilde{\zeta}_K$ are enough to deduce the leading order asymptotics of $V_K$.

Since the function $\widetilde{\zeta}_K(s)$ is meromorphic in the half-plane $\mathrm{Re}(s) > 0$, with simple poles at all the $\omega$ such that $1 - 2\ell^\omega - (n-1)r^\omega = 0$, we would deduce the following formula:

$$V_K^{[k]}(t) = \sum_\omega \frac{t^{2-\omega+k}}{(2-\omega+1)_k} \mathrm{Res}(\widetilde{\zeta}_K; \omega) + O(t^{2-\varepsilon+k}), \quad t \to 0^+.$$

Here, $\varepsilon$ denotes a small positive number, and the pointwise validity for a given value of $k$ depends on the exact growth estimates.

[5] Namely, growth conditions known as languidity of $\widetilde{\zeta}_K$; see [LRŽ17]. We shall assume here that $\widetilde{\zeta}_K$ satisfies such conditions.

# End of Presentation

Thank you for listening!

# References I


Michel L. Lapidus and Erin P. J. Pearse, *A tube formula for the Koch snowflake curve, with applications to complex dimensions*, J. London Math. Soc. (2) **74** (2006), no. 2, 397–414. MR 2269586

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Michel L. Lapidus and Machiel van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions*, second ed., Springer Monographs in Mathematics, Springer, New York, 2013, Geometry and spectra of fractal strings. MR 2977849

M van den Berg, *Renewal equation for the heat equation of an arithmetic von koch snowflake*, Infinite Dimensional Stochastic Analysis. Verh. Afd. Natuurkd **1** (1951), 2537.

# References II


Michiel van den Berg and Peter B Gilkey, *A comparison estimate for the heat equation with an application to the heat content of the s-adic von koch snowflake*, Bulletin of the London Mathematical Society **30** (1998), no. 4, 404–412.

# Appendix: Navigation Shortcuts

# Geometry of the Wedge

The contribution from $V_W$ comes entirely from a wedge arising from an angle depending only on $n$.

$$\theta = \frac{1}{2}\theta_{\text{Int}} = \frac{\pi}{2} - \frac{\pi}{n}$$

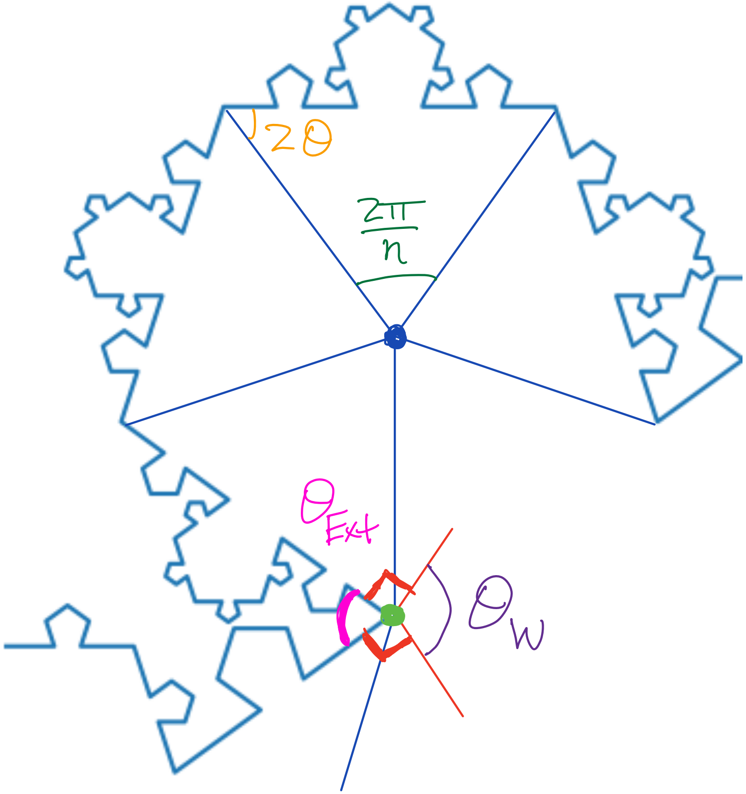
$\theta_W$ solves:

$$2\frac{\pi}{2} + \theta_W + \theta_{\text{Ext}} = 2\pi$$

$$\theta_W = \pi - (\pi - 2\theta) = 2\theta$$

Thus we have:

$$V_W(\varepsilon) = 2\theta\varepsilon^2$$



# The AFE as a Renewal Equation

A renewal equation is written in an additive form:

$$f(x) = C_1 f(x - a_1) + C_2 f(x - a_2) + \varphi(x)$$

One can solve such equations using the inverse Laplace transform to convert shifts to multiplication by exponentials.

Under appropriate conditions, such as those in [LV96], the solution takes the form:

$$f(x) = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n} c_{i_1} \cdots c_{i_n} \varphi(x - a_{i_1} - \ldots - a_{i_n})$$

# Analyticity of the Remainder

To determine the poles (and their multiplicites) of $\widetilde{\zeta}_K$, we wish to determine if the remainder integral has any singularities or not.

Based on the estimate that $R(t) \leq C_0 t^2$, we find:

$$\left| \int_0^\delta t^{s-3} R(t) dt \right| \leq \int_0^\delta t^{\sigma-1} dt = \frac{1}{\sigma} \delta^\sigma, \text{ provided } \sigma = \mathrm{Re}(s) > 0.$$

Thus, the integral is nonsingular in the right half-plane.