Explicit Formulae in Number Theory

Analytic Number Theory Course Presentation, Fall 2021

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Counting Primes

Motivating Question

How many primes are there in the first n natural numbers?

Define the prime counting function $\pi(x)$ by:

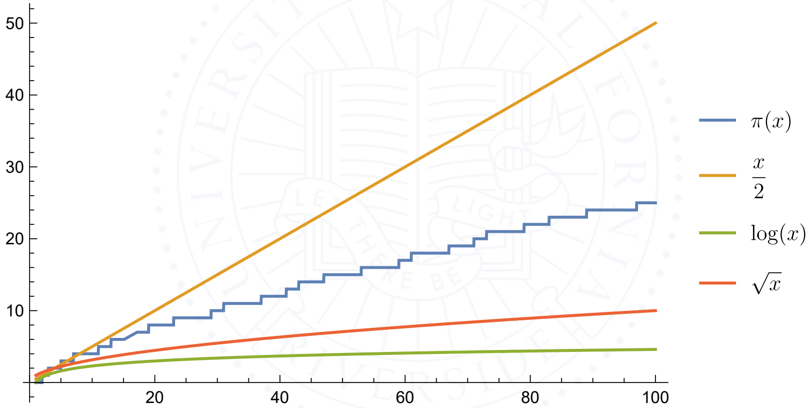
$$\pi(x) := \sum_{p \leq x} 1 = \# \{\text{primes } p : p \leq x\}$$

Reformulation

What are the asymptotics of $\pi(x)$ as $x \to \infty$?

Growth of the Prime Counting Function?

Below is the graph of the prime counting function, with some (wrong) guesses for its growth rate.



Prime Number Theorem Preliminaries

Asymptotic Equivalence

We say $f \sim g$ as $x \to \infty$ exactly when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

The logarithmic integral $\mathrm{Li}(x)$ is defined by:

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$$

The leading order behavior of $\mathrm{Li}(x)$ is given by:

$$\mathrm{Li}(x) \sim \frac{x}{\log x}, \quad x \to \infty$$

The Prime Number Theorem

Theorem [Had96, dlVP96]

Let $\pi(x)$ denote the prime counting function. Then the following holds:

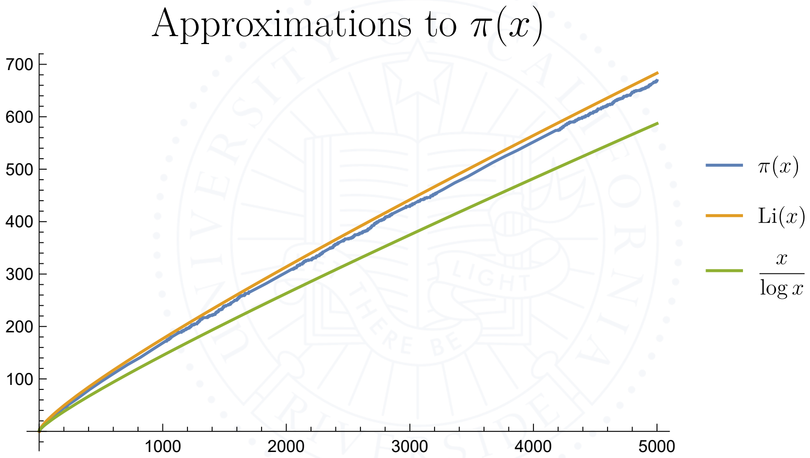
$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty$$

A better approximation is given by the following:

$$\pi(x) \sim \mathrm{Li}(x), \quad x \to \infty$$

Both proofs rely on the nonexistence of zeroes of the Riemann zeta function with real part one.

Prime Number Theorem, Visualized



Counting Prime Powers

It turns out, it is easier to study the asymptotics when we count prime powers with certain weights.

Chebyshev ψ Function [Edw74]

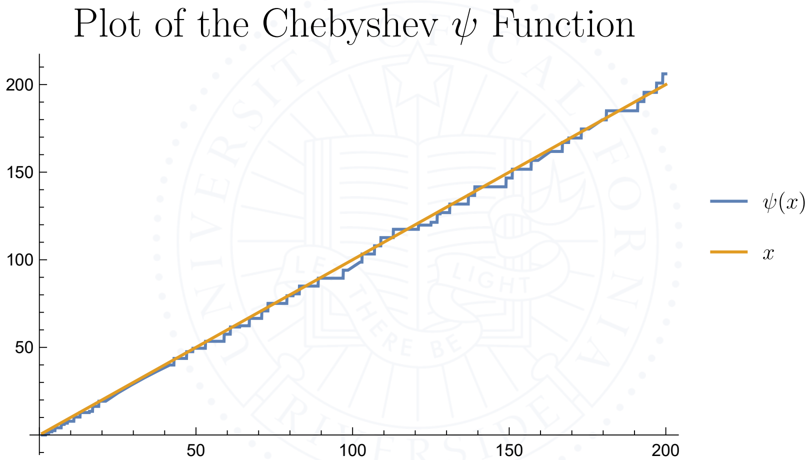
The Chebyshev ψ function counts the number of prime powers p^n less than or equal to x with the weight $\log p$:

$$\psi(x) := \sum_{p^n \leq x} \log p$$

The prime number theorem is equivalent to the following. (See for instance [Edw74, HW08].)

$$\psi(x) \sim x, \quad x \to \infty$$

Chebyshev ψ Function, Visualized



The Von Mangoldt Function and ψ_0

Recall that the von Mangoldt function is given by:

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ is a prime power} \\ 0, & \text{otherwise} \end{cases}$$

Using this, we may rewrite ψ as the following:

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

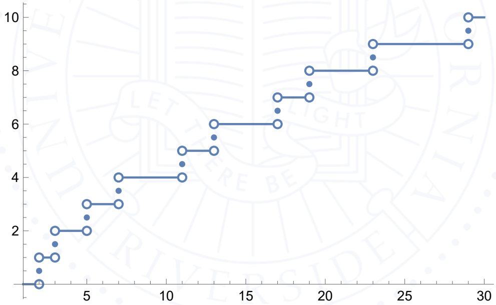
It will be useful to normalize the ψ function. Define:

$$\psi_0(x) := {\sum_{n \leq x}}' \Lambda(n) := \frac{1}{2}\left(\sum_{n < x} \Lambda(n) + \sum_{n \leq x} \Lambda(n) \right)$$

Normalized Counting Functions, Visualized

A normalized counting function takes the average value of lateral limits at jump discontinuities.

$$\pi_0(x) = \lim_{h \to 0} \frac{\pi(x+h) + \pi(x-h)}{2}$$



The Riemann-von Mangoldt Explicit Formula

Let $\zeta(s)$ denote the Riemann zeta function.

The Riemann-von Mangoldt Explicit Formula [vM95, Edw74]

Let ρ denote a critical zero of the Riemann zeta function. Then we have that:

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \log(2\pi)$$

The sum over zeroes is conditionally convergent and is ordered by increasing imaginary parts of the zeroes:

$$\sum_{\rho} \frac{x^\rho}{\rho} := \lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|\leq T} \frac{x^\rho}{\rho}$$

Perron's Formula

Consider an arithmetic function $a(n)$ with corresponding Dirichlet series:

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

Suppose f is uniformly convergent when $\mathrm{Re}(s) > \sigma_0$ and let $c > \max\{0, \sigma_0\}$ & $x > 0$.

The normalized counting function of the sequence then satisfies:

Perron's Formula [Tit86, Apo76]

$$N_a(x) = {\sum_{n \leq x}}' a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} ds$$

Residue Theorem

Let f be a function holomorphic in (a neighborhood of) a simply connected domain U except for a finite set $\Omega = \{\omega_1, ..., \omega_n\}$ of singularities.

Suppose that γ is a simple closed curve parameterizing ∂U.

Residue Theorem [Ahl78, Con78, Pal91]

$$\frac{1}{2\pi i}\oint_{\gamma} f = \sum_{\omega\in\Omega}\oint_{\partial B_{\varepsilon}(\omega)} f = \sum_{\omega\in\Omega}\operatorname{Res}(f;\omega)$$

In the above, ε is a positive real number small enough so that f is holomorphic in the annulus $B_{\varepsilon}(\omega)\setminus\{\omega\}$.

Proof Strategy

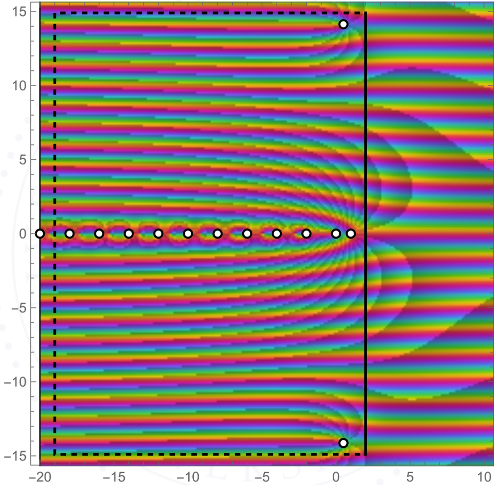
Let f be the Dirichlet series of Λ:

$$f(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Using Perron's formula and the residue theorem, we write:

$$\begin{aligned}
\psi_0(x) &= \frac{1}{2\pi i} \int_{c-iT_n}^{c+iT_n} f(s)\frac{x^s}{s} ds + R_0(x, T_n) \\
&= \sum_{\omega \in U_n} \mathrm{Res}(f(s)\frac{x^s}{s}; \omega) + \frac{1}{2\pi i} \sum_{k=1}^{3} \int_{\gamma_k} f(s)\frac{x^s}{s} ds + R_0(x, T_n) \\
&= \sum_{\omega \in U_n} \mathrm{Res}(f(s)\frac{x^s}{s}; \omega) + R_1(x, T_n, \gamma_k)
\end{aligned}$$

Proof Strategy, Visualized



The von Mangoldt Dirichlet Series

The function f satisfies the following identity:

$$f(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{d}{ds} \log \zeta(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

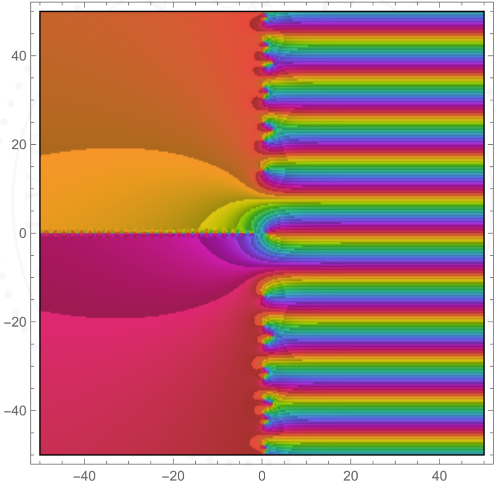
ζ has an absolutely convergent Euler product when $\mathrm{Re}(s) > 1$, allowing us to calculate:

$$\begin{aligned} -\frac{d}{ds} \log \zeta(s) &= \sum_p \log p \frac{p^{-s}}{1 - p^{-s}} \\ &= \sum_p \sum_{m=1}^{\infty} \log p (p^m)^{-s} \\ &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \end{aligned}$$

Full Proof

The Logarithmic Derivative of ζ

The function $-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s)$ has a simple pole at the pole $s = 1$ of ζ and at each zero of ζ.



Rewriting the Sum over Residues I

The singularities ω of $g(s) = -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}$ come from:

- Critical zeroes ρ of ζ
- Trivial zeroes $-2n$ of ζ
- The pole $s = 0$ of $\frac{x^s}{s}$
- The pole $s = 1$ of ζ

Note that $-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi) \neq 0$. Also, all of the trivial zeroes $-2n$ are known to be simple.

Given two functions f and h, where f has a simple pole at ω and h is holomorphic there, we can rewrite the residue:

$$\text{Res}(f(s)h(s); \omega) = h(\omega)\text{Res}(f(s); \omega)$$

Rewriting the Sum over Residues II

Further, the function $\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log\zeta(s)$ has simple poles, with residue equal to the multiplicity of the zero or pole of ζ. [1]

Using these facts, we may rewrite the sum over residues:

$$\sum_{\omega}\text{Res}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s};\omega\right) = \sum_{\omega}\text{Res}\left(g;\omega\right)$$

$$= \text{Res}(g;1) + \sum_{\rho}\text{Res}(g;\rho) + \sum_{n=1}^{\infty}\text{Res}(g;-2n) + \text{Res}(g;0)$$

$$= \frac{x^1}{1} - \sum_{\rho}\frac{x^\rho}{\rho} - \sum_{n=1}^{\infty}\frac{x^{-2n}}{-2n} - \frac{\zeta'(0)}{\zeta(0)}$$

[1] It is conjectured that all of the zeroes of ζ are simple, in which case the residue is one. If not, we take the convention that the sum over ρ repeats zeroes according to multiplicity.

Rewriting the Sum over Residues III

The third sum can be rewritten as:

$$-\sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(x^{-2})^n}{n} = -\frac{1}{2}\log(1-x^{-2})$$

So, assuming that the remainders can be appropriately estimated, we would find:

$$\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log(1-x^{-2}) - \log(2\pi)$$

The conditionally convergent sum over ρ may be defined as:

$$\sum_{\rho} \frac{x^{\rho}}{\rho} := \lim_{T_n \to \infty} \sum_{|\mathrm{Im}(\rho)| \le T_n} \mathrm{mult}(\rho)\frac{x^{\rho}}{\rho}$$

Proof Summary

The proof of the explicit formula (EF) takes the following steps:

1. Apply Perron's Formula to ψ_0
2. Equate the von Mangoldt Dirichlet series to $-\frac{\zeta'(s)}{\zeta(s)}$
3. Use the residue theorem to relate the vertical contour integral to the pole and zeroes of ζ
4. Estimate the remainder term's integration contours in the limit as the height of the horiztonal contours T_n approaches infinity

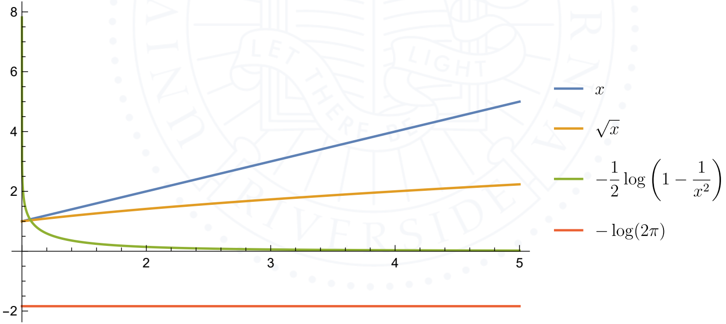
We do not comment on the integral estimates for the sake of time; see chapter three of [Edw74].

Interpretation of the Explicit Formula

Riemann-von Mangoldt Explicit Formula [vM95]

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \log(2\pi)$$

$\psi_0(x) = x(1 + o(1))$ (PNT) if and only if $\nexists \rho$ with $\mathrm{Re}(\rho) = 1$



Riemann's Original Explicit Formula [Rie59, Edw74]

Define the function:

$$J(x) := \sum_{p^n \leq x}{}' \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \pi_0(x^{\frac{1}{n}})$$

Riemann's original explicit formula takes the form:[2]

$$J(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} + \log \frac{1}{2}$$

The formula for π_0 is found by Möbius inversion:

$$\pi_0(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$$

[2] Here, $\mathrm{li}(x)$ denotes the logarithmic integral starting from lower bound zero and is a Cauchy principle value at the singularity of the integrand.

End of Presentation

Thank you for listening!

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