Resurgence in Fractal Geometry

Oral Examination, Winter 2021

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Doctoral Thesis/Dissertation Proposal

My Project, Preliminary Description

I intend to apply tools and techniques from resurgent asymptotics to fractal geometry. My main focus is on empowering/better understanding explicit formulae that relate geometric (or spectral) functions to poles of an associated zeta function, looking for situations where new phenomena might manifest.

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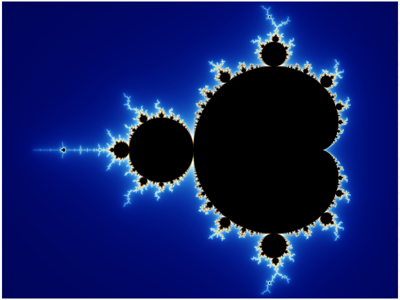
To understand this better, we shall discuss:

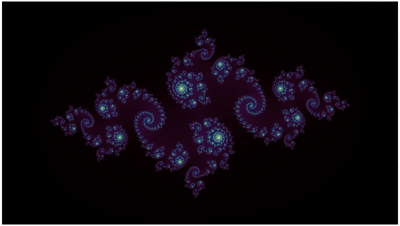
- Fractal Geometry
- Explicit Formulae
- Resurgent Asymptotics

Fractal Geometry

Navigation Shortcuts

Fractals





Fractal Geometry and Geometric Oscillations

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Properties of the fractal can be expressed in terms of these complex dimensions. One such property would be the volume of the set of points within a certain distance from the fractal.

Example: The Cantor Set

The standard middle-thirds Cantor set:

At the n^{th} stage, 2^{n-1} intervals of length 3^{-n} are removed. The Cantor string $\mathcal{L}_{CS} = \{\ell_n\}_{n\in\mathbb{N}}$ is then a multiset of these lengths, repeated according to multiplicity: $\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \ldots\right\}$

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The fractal zeta function ζ_{CS} is given by:

$$\zeta_{CS}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{1}{3^s - 2}$$

Ordinary Fractal String as a Measure

An ordinary fractal string $\mathcal{L} = \{\ell_n\}_{n\in\mathbb{N}}$ may be represented as a measure: [1]

$$\mu_{\mathcal{L}} = \sum_{j=1}^{\infty} \delta_{\{\ell_j^{-1}\}}$$

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This construction works for any sufficiently nice measure, not just those from fractal strings.

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Generalized Fractal String

Definition

A **generalized fractal string** is a local positive or complex measure η defined on $(0, \infty)$.[2] We also stipulate that η has no mass near zero, i.e. there exists a positive number x_0 for which $|\eta|[(0, x_0)] = 0$, where $|\eta|$ denotes the variation of η.

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More on the Counting Function

Ordinary Counting Function

The **geometric counting function** of an ordinary fractal string $\mathcal{L}$:

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For a general measure η, we may write:

$$N_\eta(x) = \int_0^x d\eta = \eta\big((0,x)\big) + \frac{1}{2}\eta(\{x\})$$

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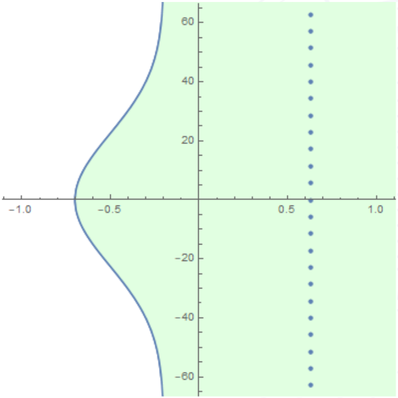
Golden String, $\mathcal{L}_{GS} = \left\{1, \frac{1}{2}, \frac{1}{2^\varphi}, \frac{1}{4}, \frac{1}{2 \cdot 2^\varphi}, \frac{1}{2 \cdot 2^\varphi}, \frac{1}{2^\varphi \cdot 2^\varphi}, ...\right\}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

$$\zeta_{GS}(\omega) = \frac{1}{1 - 2^{-\omega} - 2^{-\varphi\omega}} = \infty \iff 2^{-\omega} + 2^{-\varphi\omega} = 1$$

Complex Dimensions Plotted

The Cantor String (Screened)



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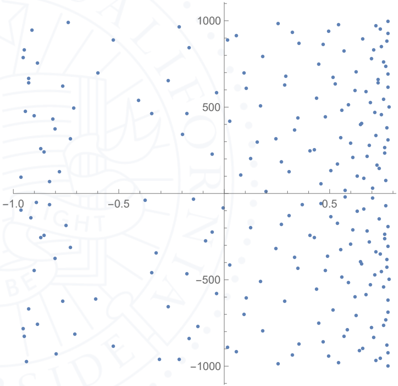

Explicit Fomulae

Navigation Shortcuts

Riemann Zeroes & the Chebyshev ψ Function

Let $\psi(x)$ count the number of prime powers p^m less than or equal to x with weight $\log p$, and suppose it is normalized at jump discontinutities. We may write $\eta = \sum_{p^m} \log p \, \delta_{\{p^m\}}$.

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Then $\zeta_\eta(s) = \int_0^\infty \psi(x) x^{-s} dx = -\frac{\zeta'(s)}{\zeta(s)}$, and further:

$$\begin{aligned}
\psi(x) &= \sum_{p \leq x} k \log p \quad \text{ where } p^k \leq x < p^{k+1} \\
&= \sum_{\omega} \frac{x^\omega}{\omega} \mathrm{Res}(\zeta_\eta(s); \omega) - \frac{\zeta'(0)}{\zeta(0)} \\
&= \frac{x^1}{1} - \sum_{\rho} \frac{x^\rho}{\rho} - \sum_{k=1}^{\infty} \frac{x^{-2k}}{-2k} - \log 2\pi \\
&= x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2} \log(1 - x^2) - \log(2\pi)
\end{aligned}$$

Explicit Formula via Complex Dimensions

Pointwise E.F., with Error (Thm 5.10 in [LvF13])

Let η be a *languid* generalized fractal string, k a sufficiently large positive integer, [3] and $D_\eta(W)$ the visible complex fractal dimensions of η in the window W to the right of screen S. Then for all $x > 0$,

$$
\begin{aligned}
N_\eta^{[k]}(x) &= \sum_{\omega \in D_\eta(W)} \operatorname{Res}\left(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega\right) \\
&\quad + \frac{1}{(k-1)!} \sum_{\substack{j=0 \\ -j \in W \setminus D_\eta}}^{k-1} \binom{k-1}{j} (-1)^j x^{k-1-j} \zeta_\eta(-j) \\
&\quad + O\left(x^{\sup \operatorname{Re}(S)+k-1}\right)
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&+ O\left(x^{\sup \operatorname{Re}(S) + k - 1} \right)
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The sum is understood to be ordered by increasing magnitude of the imaginary part of the poles ω. $k = 1$ corresponds to the usual counting function.

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Explicit Formula Notes

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- The remainder term is an integral over the screen, with the integrand the same as the function inside the residue.
- Higher dimensional formulae exist.

Resurgent Asymptotics

Navigation Shortcuts

Asymptotic Expansions

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The following are equivalent definitions:

$$f(z) \sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad z \to \infty$$

$$f(z) = \sum_{n=1}^{N} \frac{a_n}{z^n} + O\left(\frac{1}{z^{N+1}}\right), \quad z \to \infty, \forall N \in \mathbb{N}$$

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Asymptotic Expansion Examples

Stirling's series:

$$\log(\Gamma(x)) \sim \left(x - \tfrac{1}{2}\right)\log(x) - x + \frac{1}{2}\log(2\pi) + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)} x^{-2j+1}, \quad x \to \infty$$

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Sine and an exponentially small term:

$$\sin(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \sim \sin(z) + e^{-1/z}, \quad z \to 0^+$$

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Euler and the exponential integral:

$$-e^x \mathrm{Ei}(-x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{x^{k+1}}, \quad x \to +\infty$$

Stokes Phenomenon

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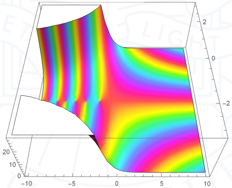
Transseries are a broader class of series that can contain all of the important terms. We make sense of them via stronger Borel resummation techniques.

Supernumerary Bows & The Airy Function

The Airy function, a solution to $y'' - xy = 0$, has two different asymptotic expansions. To first order, we have:

$$\mathrm{Ai}(z)$$

(Entire)



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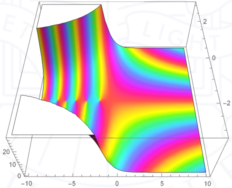
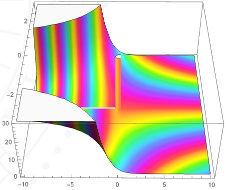
$\mathrm{Ai}(z)$	$\dfrac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$
(Entire)	$\lvert \arg(z) \rvert < \pi$





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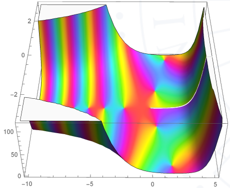

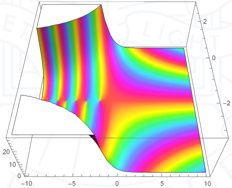

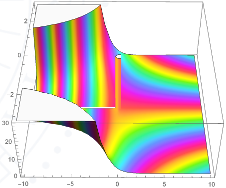

Airy Function Expansion

The Airy function is governed by the asymptotic expansion:

$$\varphi_{\mathrm{Ai}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} \frac{1}{z^n}$$

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More remarks:

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- $z = k^{\frac{3}{2}}$ is a natural change of variables for ensuing resummation.

Borel Summation

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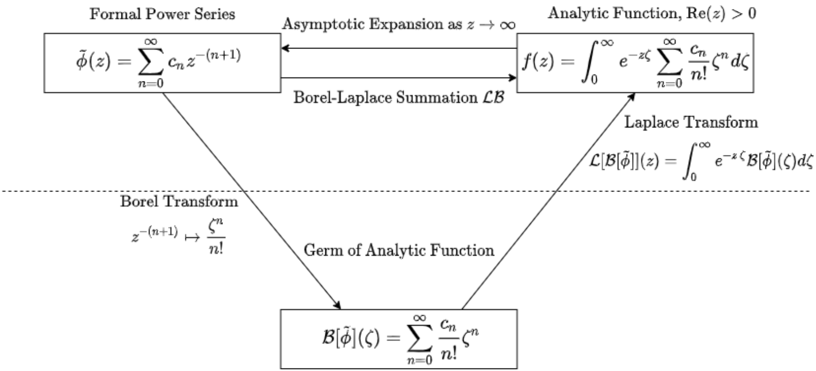
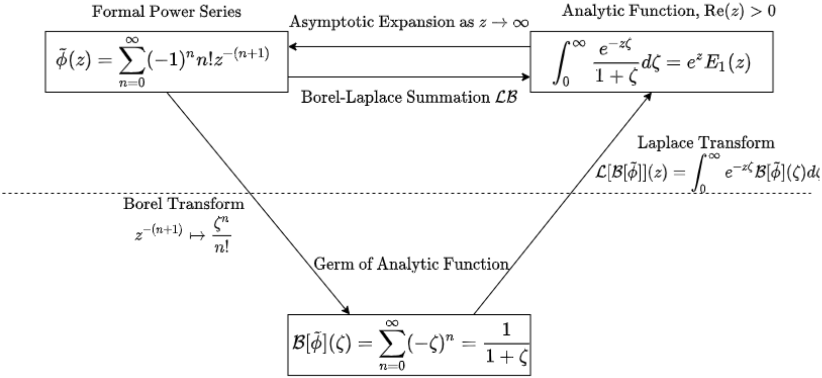
Key Steps:

- Borel Transform
- Analytic Continuation in the Borel Plane
- Dealing with Singularities
- Laplace Transform Back

Borel Summation: Schematic



Borel Summation: Example



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Example with the Airy function resummation.

Transsseries Short Introduction

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Transseries Short Introduction

Transseries are elements of a (formal) differential field obtained by extending/closing ordinary series under additional algebraic and differential operations.

- They have an ordering akin to asymptotic growth rate.
- Summable transseries are those that correspond, under enhanced Borel resummation, to bona fide functions.
- Examples:

$$-4\exp\left(\sum_{j=0}^{\infty} x^{-j}e^{x}\right) + \sum_{j=0}^{\infty} x^{-j}e^{x} - 17 + \pi x^{-1}$$

$$\exp\left((\log\log x)^{\frac{1}{2}}\right) + (\log\log x)^{\frac{1}{2}} + x^{-2}$$

Transseries Schematic

$$
\begin{tikzcd}
\text{Formal Power Series} \arrow[d] & \text{Summable Power Series} \arrow[l, "\text{Embedding}"'] \arrow[r, shift left, "\text{Resummation}"] \arrow[d, "\text{Closure under } (+,\times,\circ,\partial) \text{ and } (+,\times,\circ,\partial)^{-1}"] & \text{Analytic Germs} \arrow[l, shift left] \arrow[d] \\
\text{Formal Transseries} & \text{Summable Transseries} \arrow[l] \arrow[r, shift left] & \text{Analyzable Germs} \arrow[l, shift left, "\text{Formalization}"]
\end{tikzcd}
$$

Resurgent Functions

Our (Particular Type of) Resurgent Functions

Resurgent functions include formal asymptotic power series (in reciprocal powers of x) whose Borel transforms correspond to germs of analytic functions that can be analytically continued in the Borel plane.

Roughly speaking, resurgent functions are those with the necessary conditions for which enhanced Borel resummation will succeed. One can consider them to be: the formal objects prior to resummation, the corresponding analytic germs obtained after the Borel transform, or the resummed/bona fide functions.

Resurgence & Alien Derivatives

An operator called the alien derivative connects the behavior of a resurgent function near the origin to behavior near other singular points.

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J. Écalle on the namesake “Resurgence”

[Alien derivatives] enable us to describe, by means of so-called resurgence equations... the very close connection which usually exists between the behavior of [the Borel transformed analytic germ] near $0_\bullet$ and near its other singular points ω.

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This self-reproduction property is an outstanding feature of all resurgent functions of natural origin (their birth-mark, as it were!) and it is precisely what the label "resurgence" (bestowed somewhat promiscuously on the whole algebra [of such functions]) is meant to convey.

Doctoral Thesis/Dissertation Project, Restated

My Project Description, More Precisely

I intend to study explicit formulae (that admit analytic continuation in the complex plane,) and to determine where and why their asymptotics may change (cf. Stokes phenomena.)

In such scenarios, I aim to find more complete descriptions, or any "missing" terms that could become relevant.

To achieve these goals, I intend to use enhanced Borel summation, transseries, and other tools in resurgent analysis to study relevant expansions as inputs/parameters change.

Known Applications of Explicit Formulae and of Resurgence

Navigation Shortcuts

Motiviation & Relevance

Motivating idea from mathematical physics:

- Recovering non-perturbative effects from perturbative asymptotic expansions

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Known uses of resurgence and of explicit formulae:

- Explicit Formulae & Proving the Prime Number Theorem
- Inverse Spectral Problem & The Riemann Hypothesis
- Fractal Tube Formulae & Higher Dimensions
- Applications of Resurgence
- Resurgent Analysis & Fractal Geometry

End of Presentation

Thank you for listening!

Appendix: Navigation Shortcuts

Explicit Formulae & Proof of the Prime Number Theorem

A Formula for the Riemann Zeta Function

Let ζ be the Riemann zeta function; it is strongly languid with $k = 0$ and $A = 1$. Denote by $\mathcal{P} = \sum_{m \geq 1, p} (\log p) \delta_{\{p_m\}}$ the geometric zeta function of the prime string. Then for all $x > 1$, (in a distributional sense,)

$$\mathcal{P} = 1 - \sum_{\rho} x^{\rho - 1} + \sum_{n=1}^{\infty} x^{-(2n+1)}$$

This formula can be used to derive the following formula for the prime counting function π, and thus the prime number theorem.

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-c\sqrt{\log x}})$$

Return to applications overview.

Inverse Spectral Problem & The Riemann Hypothesis

Inverse Spectral Problem with Fractal Strings $(\mathrm{ISP})_D$

Let $\mathscr{L}$ be a fractal string corresponding to the set Ω and having Minkowski dimension $D \in (0, 1)$. We say that $(\mathrm{ISP})_D$ holds if, for any such fractal string $\mathscr{L}$, whenever $N_{\mathscr{L}}(\lambda)$ asymptotically has a monotonic second term, then the boundary $\partial\Omega$ is Minkowski measurable.

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Theorem (Lapidus and Maier, 1995)

$(\mathrm{ISP})_D$ holds for a given value of $D \in (0,1)$ iff the Riemann zeta function does not have any zeroes on the vertical line $\mathrm{Re}(s) = D$. As a corollary, the Riemann hypothesis holds iff $(\mathrm{ISP})_D$ holds exactly for all $D \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Return to applications overview.

Major Uses of Resurgent Analysis

Dulac's Conjecture

- On finiteness of limit cycles; related to Hilbert's 16^{th} problem
- Écalle's proof pioneers resurgent analysis

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Quantum Field Theory

- Exponentially small, non-analytic corrections to perturbative expansions ("instantons")
- Potential to recovering nonperturbative effects through resurgence of a perturbative expansion

More Resurgence Applications in Mathematical Physics

- Normal forms of dynamical systems
- Gauge theory of singular connections
- Quantization of symplectic and Poisson manifolds
- Floer homology and Fukaya categories
- Knot invariants
- Wall-crossing and stability conditions in algebraic geometry
- Spectral networks
- WKB approximation in quantum mechanics
- Non-linear differential equations and asymptotics

Return to applications overview.

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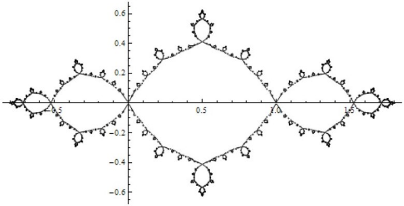
- Transseries formulae have been useful in describing some Julia sets.



Resurgent Analysis & Fractal Geometry

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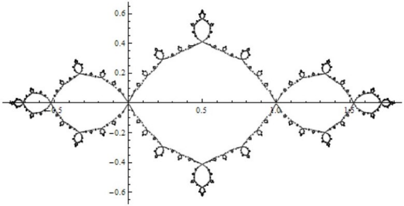

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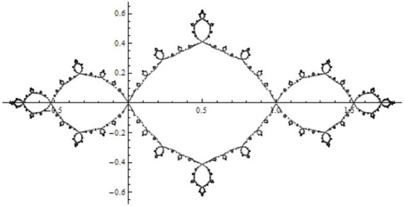

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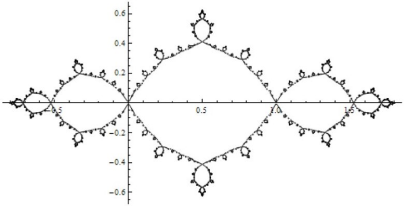

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Return to applications overview.

Appendix: Ordinary Fractal String/Zeta Function

Let Ω be a bounded open set in $\mathbb{R}$; then Ω may be written as a countable union of disjoint open intervals.

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The zeta function associated to $\mathcal{L}$ is given by:

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s$$

Return to fractal strings and zeta functions from measures.

Appendix: Languid Growth Conditions

- Let S denote the screen for ζ_η, viz. $S = \{s(t) + it : t \in \mathbb{R}\}$.

Return to pointwise explicit formula with error term.

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Polynomial growth on a sequence of horizontal lines (L1)

$$\forall n \in \mathbb{Z}, \forall \sigma \geq s(T_n), \quad |\zeta_\eta(\sigma + iT_n)| \leq C(|T_n| + 1)^\kappa$$

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$$\forall n \in \mathbb{Z}, \forall \sigma \geq s(T_n), \quad |\zeta_\eta(\sigma + iT_n)| \leq C(|T_n| + 1)^\kappa$$

Polynomial growth along the given screen (L2)

$$\forall t \in \mathbb{R}, |t| \geq 1, \quad |\zeta_\eta(s(t) + it)| \leq |t|^\kappa$$

Return to pointwise explicit formula with error term.

Appendix: Higher Dimensional Tube Formulae

Relative Tube Function, and antiderivatives:

$$V_{A,\Omega}^{[0]}(t) := |A_t \cap \Omega|, \quad V_{A,\Omega}^{[k]}(t) := \int_0^t V_{A,\Omega}^{[k-1]}(\tau)d\tau$$

Relative Tube Zeta Function, for (A, Ω) in $\mathbb{R}^N$:

$$\tilde{\zeta}_{A,\Omega}(s) := \int_0^\delta t^{s-N-1}|A_t \cap \Omega|dt$$

Pointwise Tube Formula with Error, for languid relative fractal drums:

$$V_{A,\Omega}^{[k]}(t) = \sum_{\omega \in \mathscr{P}(\tilde{\zeta}_{A,\Omega}, W)} \operatorname{Res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\tilde{\zeta}_{A,\Omega}(s); \omega\right) + \tilde{R}_{A,\Omega}^{[k]}(t)$$

Return to one dimensional formula.

Appendix: Minkowski Dimension and Measurability

Let $A \subseteq \mathbb{R}^n$.
Upper Minkowski Content:

$$\mathscr{M}_d^*(A) := \limsup_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{\varepsilon^{n-d}}$$

The lower Minkowski content $\mathscr{M}_{*,d}$ is with a limit infimum.

Minkowski Dimension D:

$$\exists! D, \text{ if } d < D, \mathscr{M}_d^*(A) = \infty \text{ and if } D < d, \mathscr{M}_d^*(A) = 0$$

Minkowski Measurability:

$$\mathscr{M}_D^* = \mathscr{M}_{*,D} =: \mathscr{M}, \text{ and } 0 < \mathscr{M} < \infty$$

Return to ISP and RH.

Airy Series: Borel Summation

- The minor of φ_{Ai} is its (formal) Borel transform, forgetting the constant term:

$$\tilde{\varphi}_{\text{Ai}} := \mathcal{B}[\varphi_{\text{Ai}}] = \sum_{n=1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$$

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- $\tilde{\varphi}_{\mathrm{Ai}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \left\{0, -\frac{4}{3}\right\}$
- For any direction θ not along the negative real axis, the following converges for $\mathrm{Re}(ze^{i\theta}) > 0$:

$$S_\theta \varphi_{\mathrm{Ai}}(z) := a_0 + \mathcal{L}_\theta \mathcal{B}[\varphi_{\mathrm{Ai}}](z) = a_0 + \int_0^{\infty e^{i\theta}} \tilde{\varphi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} d\zeta$$

Airy Function, Resummation on $\mathbb{R}^+$

Where before:

$$\mathrm{Ai}(k) \sim \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

We now have:

$$\mathrm{Ai}(k) = \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} S_0 \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

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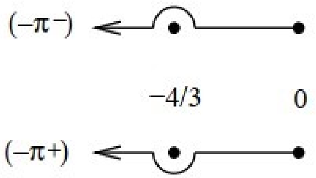
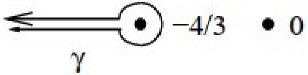
This resummation is valid for $|\arg(k)| < \frac{\pi}{3}$, $|k| > 0$.

One can rotate the direction of summation for new regions of validity. See: Airy function on the negative real line.

Airy Function Resummation near $\theta = -\pi$

Contours above and below the singularity:





Relationship of the resummations:

$$S_{-\pi^-}\varphi_{\mathrm{Ai}}(z) = S_{-\pi^+}\varphi_{\mathrm{Ai}}(z) + \int_\gamma \widetilde{\varphi_{\mathrm{Ai}}}(\zeta)e^{-z\zeta}d\zeta$$

Alien Calculus & Behavior across the Singularity

The Hankel contour γ can be expressed using the so-called alien derivative:

$$\int_{\gamma} \tilde{\varphi}_{\text{Ai}}(\zeta) e^{-z\zeta} d\zeta = e^{+\frac{4}{3}z} S_{-\pi} \left(\Delta^{z}_{-\frac{4}{3}} \varphi_{\text{Ai}} \right)(z)$$

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In this case,

$$\Delta^z_{-\frac{4}{3}} \varphi_{\mathrm{Ai}} = -i\varphi_{\mathrm{Bi}}, \quad \varphi_{\mathrm{Bi}}(z) := \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{z^n}$$

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φ_{Bi} is also Gevrey-1 and its minor $\tilde{\varphi}_{\mathrm{Bi}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \left\{0, +\frac{4}{3}\right\}$.

Return to resummation on $\mathbb{R}^+$. More on the Airy function resummed on $\mathbb{R}^-$.

Appendix: Airy Function on $\mathbb{R}^-$

Deducing the behavior Ai for negative real inputs.

Appendix: Airy Function on $\mathbb{R}^-$

Deducing the behavior Ai for negative real inputs.

Airy expansion when $|\arg(k) - \pi| < \frac{\pi}{3}$, $z = k^{\frac{3}{2}}$:

$$\mathrm{Ai}(k) = \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} \left(e^{-\frac{2}{3}z} S_{-\frac{3\pi}{2}} \varphi_{\mathrm{Ai}}(z) + i e^{+\frac{2}{3}z} S_{-\frac{3\pi}{2}} \varphi_{\mathrm{Bi}}(z) \right)$$

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Note the new exponential term that appeared.

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One can rewrite the LHS as the resummed version of the second expansion we saw previously.

Return to Airy Resummation.

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